Symmetry in Euclidean Space

Definitions Given $n \in \mathbb{N}$,

n-dimension Euclidean space $= \mathbb{R}^n$ equipped with <u>standard inner product</u> (scalar/dot product)

$$\left\langle \underbrace{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}_{=\underline{x}}, \underbrace{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}}_{=\underline{y}} \right\rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

The <u>distance</u> between $\underline{x}, \underline{y} \in \mathbb{R}^n$. Same as usual concept in $\mathbb{R}, \mathbb{R}^2, \mathbb{R}^3$.

$$d(\underline{x}, \underline{y}) := \sqrt{\langle \underline{x} - \underline{y}, \underline{x} - \underline{y} \rangle}$$

Definition An <u>isometry</u> of $\mathbb{R}^n$ is a map $f : \mathbb{R}^n \to \mathbb{R}^n$ (not necessarily linear) such that $d(f(\underline{x}), f(\underline{y})) = d(\underline{x}, \underline{y}), \ \forall \underline{x}, \underline{y} \in \mathbb{R}^n$

preserves distances

The set of all isometries of $\mathbb{R}^n$ is denoted <u>$\mathrm{Isom}(\mathbb{R}^n)$</u>.

Remarks

- $\mathrm{Id}_{\mathbb{R}^n} \in \mathrm{Isom}(\mathbb{R}^n)$ and $f, g \in \mathrm{Isom}(\mathbb{R}^n) \Rightarrow f \circ g \in \mathrm{Isom}(\mathbb{R}^n)$
- $f \in \mathrm{Isom}(\mathbb{R}^n)$ <u>and</u> $f(\underline{0}) = \underline{0} \iff f(\underline{x}) = A\underline{x}$ for $A \in O_n(\mathbb{R})$ — orthogonal $n \times n$ matrices ($A^T = A^{-1}$)

For example $f(\underline{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \underline{x}$ — anticlockwise rotation by θ

- $f \in \mathrm{Isom}(\mathbb{R}^n)$ a translation $\iff \exists\, \underline{y} \in \mathbb{R}^n$ such that $f(\underline{x}) = \underline{x} + \underline{y}$ $\forall \underline{x} \in \mathbb{R}^n$
- $f \in \mathrm{Isom}(\mathbb{R}^n) \iff f(\underline{x}) = \underbrace{A\underline{x}}_{\text{fixes } \underline{0}} + \underbrace{\underline{y}}_{\text{translation}}$ for some $A \in O_n(\mathbb{R}), \underline{y} \in \mathbb{R}^n$

Definition Let $X \subset \mathbb{R}^n$ be a <u>subset</u>.

$$\mathrm{Sym}(X) := \{ f \in \mathrm{Isom}(\mathbb{R}^n) \mid f \text{ permutes } X \}$$

Called Symmetry group of $X \subset \mathbb{R}^n$

Observation $\mathrm{Sym}(X)$ naturally acts on X.

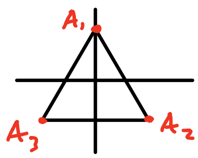
$$\mathrm{Sym}(X) \times X \longrightarrow X$$
$$(f, \underline{x}) \longmapsto f(\underline{x})$$

Dihedral Groups

$m \in \mathbb{N},\ m \geq 3$

$X_m = \{A_1, A_2, \ldots, A_m\} =$ vertices of a regular m-gon, centered at $\underline{0} \subset \mathbb{R}^2$

Example



$X_3 = \{A_1, A_2, A_3\}$

Definition

$D_m = \mathrm{Sym}(X_m) \subset \mathrm{Isom}(\mathbb{R}^2)$

m th dihedral group

$D_m \cong$ subgroup of Sym_m

Important Observation : D_m acts faithfully on $\{A_1, A_2, \ldots A_m\}$

Rot_m = Rotational symmetries

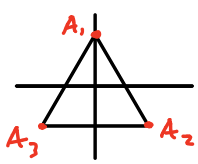
$= \{e, \sigma, \sigma^2, \ldots, \sigma^{m-1}\} \subset D_m$ subgroup

Rotation by 0 about $\underline{0}$

rotation clockwise by $\frac{2\pi}{m}$ about $\underline{0}$

cycle notation in $\Sigma(\{A_1, A_2, A_3\})$

Example



$\sigma = \frac{2\pi}{3}$ $\quad \sigma^2 = \frac{4\pi}{3}$ $\Rightarrow$ $\sigma = (A_1 A_2 A_3)$, $\sigma^2 = (A_1 A_3 A_2)$

More generally, $\sigma^d(A_1) = A_{d+1} \quad \forall\, d \in \{0, \ldots, m-1\}$

$\Rightarrow$ D_m acts transitively on $\{A_1, \ldots, A_m\}$.

Observe that given $A_i \in X_m$, $\mathrm{Stab}(A_i) = \{e, \tau_i\}$

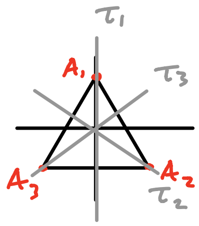
reflection in line through $\underline{0}$ and $\underline{A_i}$

$\Rightarrow \quad |D_m| = |\mathrm{Orb}(A_i)| \cdot |\mathrm{Stab}(A_i)| = 2m$

$|\mathrm{Rot}_m| = m \quad \Rightarrow \quad D_m = \mathrm{Rot}_m \amalg \underbrace{\tau\, \mathrm{Rot}_m}_{\text{All reflective symmetries}}$

any reflection

Example



$\Rightarrow \quad D_3 = \{e, \sigma, \sigma^2, \tau_1, \overset{\tau,\sigma}{\overset{\shortparallel}{\tau_2}}, \overset{\tau,\sigma^2}{\overset{\shortparallel}{\tau_3}}\}$

Rotation by $\frac{2\pi}{m}$ about $\underline{o}$

Any reflection symmetry

Hence D_m is generated by σ and τ with relations

- $\mathrm{ord}(\sigma) = m$
- $\mathrm{ord}(\tau) = 2$
- $\tau\sigma = \sigma^{-1}\tau = \sigma^{m-1}\tau$

Symmetries in $\mathbb{R}^3$

All faces equal regular m-gons

Platonic Solid in $\mathbb{R}^3$ = Regular, convex polyhedron in $\mathbb{R}^3$

Amazing Fact : There are only five possibilities

X = platonic solid, centered at $\underline{o} \subset \mathbb{R}^3 \Rightarrow \mathrm{Sym}(X) = ?$

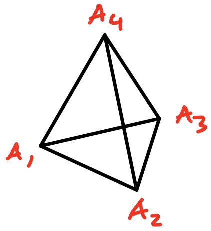
Important Observation :

$\Rightarrow$ $\mathrm{Sym}(X)$ acts faithfully $\Rightarrow$ $\mathrm{Sym}(X) \cong$ Subgroup of Σ(Vertices of X)

$\mathrm{Rot}(X)$ = rotational symmetries.

Lets just say $\mathrm{Sym}(X) \subset \Sigma(\text{Vertices of } X)$

X = Tetrahedron



$\Rightarrow \mathrm{Sym}(X) \subset \Sigma(\{A_1, A_2, A_3, A_4\})$

What permutations can show up ?

Let τ = Reflection in plane containing A_1 and A_2

$\Rightarrow \quad \tau = (A_3 A_4) \in \Sigma(\{A_1, A_2, A_3, A_4\})$.

$\Rightarrow \quad (A_i A_j) \in Sym(X) \quad \forall \; i \neq j \quad \Rightarrow \quad Sym(X) = \Sigma(\{A_1, \ldots, A_4\}) \cong Sym_4$

σ = Rotation by $\frac{2\pi}{3}$ about line containing $\underline{0}$ and A_4 $\Rightarrow$

$\sigma = (A_1 A_2 A_3) \in \Sigma(A_1, A_2, A_3, A_4)$

We could do this for any $(A_i A_j A_k)$

Cycles of length 3 generate $Alt_4 \Rightarrow Rot(X) \cong Alt_4$

<u>Amazing Fact</u> : Any finite subgroup of $Isom(\mathbb{R}^3)$ must be a subgroup of a dihedral group or $Sym(X)$, where X is a platonic solid.

<u>Even more amazing fact</u> : There are versions of platonic solids in $\mathbb{R}^n$ for $n > 3$

<u>$n = 4$</u> : There are 6. For example the hypercube (Tesseract)

5 come from $\mathbb{R}^3$

There's one unique to $\mathbb{R}^4$

<u>$n > 4$</u> : There are 3

Higher dimensional versions of tetrahedron, cube and octahedron.